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Fluctuation dissipation ratio in the one-dimensional kinetic Ising model

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The exact relation between the response function $R(t,t')$ and the two time correlation function $C(t,t')$ is derived analytically in the one-dimensional kinetic Ising model subjected to a temperature quench. The fluctuation dissipation ratio $X(t,t')$ is found to depend on time through $C(t,t')$ in the time region where scaling $C(t,t')=f(t/t')$ holds. The crossover from the nontrivial form $X[C(t,t')]$ to $X(t,t')\equiv 1$ takes place as the waiting time t_w is increased from below to above the equilibration time t_{eq}.

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I. INTRODUCTION

The time evolution of a system relaxating to equilibrium is characterized by two distinct time regimes: the off equilibrium transient for $t<t_{eq}$ and the stationary equilibrium evolution for $t>t_{eq}$, where t_{eq} is the equilibration time. Normally, t_{eq} is smaller than experimental times and one can actually observe equilibration. However, the existence of situations in which the opposite is true, namely, where t_{eq} exceeds by far any practical observation time as in glassy systems at low temperature, has contributed to arise a lot of interest in the off equilibrium relaxation regime. It turns out that quite a useful tool in the investigation of these slowly relaxating systems, with and without disorder, is the fluctuation dissipation relation.

For definiteness, let us consider a system in equilibrium at some temperature T_I. At the time $t=0$ a quench to a lower temperature T_F is performed. On top of this primary relaxation process a second one is activated by switching on an external field at the time $t_w>0$. Characterizing the response of the system under the action of the perturbation by the response function $R(t,t')$, the search for a fluctuation dissipation relation aims to connect $R(t,t')$ to the relevant correlation function $C(t,t')$ in the unperturbed relaxation process. Given $R(t,t')$ and $C(t,t')$ one can always write

$$R(t,t')=\frac{X(t,t')}{T_F}\frac{\partial}{\partial t'}C(t,t'), \tag{1}$$

where $t\geqslant t'$. Without any further specification this is just a definition of the quantity $X(t,t')$, which is called the fluctuation dissipation ratio (FDR). Equation (1) acquires predictive power when independent statements are made about the FDR. Thus, if the shortest time t' is greater than the equilibration time t_{eq}, i.e., if one looks into the time translation invariant equilibrium dynamics, the fluctuation dissipation theorem (FDT) requires $X(t,t')\equiv 1$. This is no more true when $t'<t_{eq}$. However, if appropriate conditions on t',t,t_{eq} are satisfied it may turn out that $X(t,t')$ depends on the time arguments only through $C(t,t')$. This was first discovered by Cugliandolo and Kurchan [1] in the context of mean field spin glass models at low temperature. In that case the system does not equilibrate ($t_{eq}=\infty$) and $X(t,t')=X[C(t,t')]$ in the asymptotic limit of large times. Subsequently the validity of this relation was verified for finite dimensional spin glass models [2] and also in the coarsening processes of non disordered systems [3,4]. As a matter of fact a classification of slowly relaxating systems can be made [5] on the basis of the behavior of $X(C)$.

The relation between X and C is important for different reasons. From the point of view of analytical calculations it allows us to close the equations of motion for R and C [1]. From the more fundamental point of view of the understanding of the off equilibrium dynamics it can be related to the effective temperature of different dynamical modes [6] and under certain hypothesis it provides a connection between the relaxation regime and the structure of equilibrium states [7].

In this paper we analyze the relaxation to equilibrium in the one-dimensional kinetic Ising model quenched from the initial temperature T_I to the lower final temperature T_F. The correlation length then grows from some initial value ξ_I, which we assume $O(1)$, to the final value

$$\xi_F=-[\ln\tanh(J/T_F)]^{-1} \tag{2}$$

where $J>0$ is the nearest-neighbor ferromagnetic interaction. The equilibration time is defined by

$$t_{eq}=\xi_F^2. \tag{3}$$

By lowering the temperature of the quench t_{eq} can be tuned at will with $\lim_{T_F\to 0}t_{eq}=\infty$ allowing for the investigation of the slow relaxation coming from the high-temperature side. We compute the response function $R(t,t')$ after switching on a random external field at the time t_w after the quench. We are then able to analyze in detail the changeover from the equilibrium to the off equilibrium regime by monitoring the change in the FDR (or in the integrated response) as t_w is varied from $t_w>t_{eq}$ to $t_w<t_{eq}$. When $t_w>t_{eq}$ dynamics is time translation invariant and the usual FDT holds. When the region $t_w<t_{eq}$ is entered the deviation from FDT occurs. However, if the difference between t_w and t_{eq} is sufficiently

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large, there is a range of values between t_w and $t_{\rm eq}$ where $C(t,t')$ scales. Within this range the FDR depends only on C, while outside there remains an explicit dependence on t_w. In other words with a finite but large $t_{\rm eq}$ the off equilibrium dynamics follows the pattern of interrupted aging and we find that the FDR depends only on C as long as aging holds. The case of the zero temperature quench is the limiting case ($t_{\rm eq}=\infty$) where aging occurs for arbitrarily large times yielding $X(t,t')=X[C(t,t')]$ at all times.

II. UNPERTURBED CORRELATION FUNCTIONS

In the following we consider a one-dimensional ferromagnetic Ising model with nearest-neighbor interaction:

$$\mathcal{H}[\sigma]=-J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} \tag{4}$$

evolving in time with Glauber single spin flip dynamics

$$\frac{\partial}{\partial t}P([\sigma],t)=\sum_i\{w(-\sigma_i)P([R_i\sigma],t)-w(\sigma_i)P([\sigma],t)\}, \tag{5}$$

where $P([\sigma],t)$ is the probability of realization of the configuration $[\sigma]$ at the time t, $[R_i\sigma]$ is the configuration with the ith spin reversed,

$$w(\sigma_i)=\frac{1}{2}\left[1-\frac{\gamma}{2}\sigma_i(\sigma_{i-1}+\sigma_{i+1})\right] \tag{6}$$

is the transition rate from $[\sigma]$ to $[R_i\sigma]$, and $\gamma=\tanh(2J/T_F)$.

Given an initial probability distribution $P([\sigma],t=0)$, with the choice (6) for the transition rate the solution of Eq. (5) for large time reaches the equilibrium Gibbs state $P_{\rm eq}[\sigma]=1/Z\exp[-(1/T_F)\mathcal{H}[\sigma]]$. The dynamics in the one-dimensional case has been solved by Glauber [8]. Let us summarize those properties of the time dependent correlation functions which will be needed in the following. Assuming that the initial state $P([\sigma],t=0)$ is symmetrical the probability distribution $P([\sigma],t)$ remains symmetrical throughout yielding $\langle\sigma_i(t)\rangle\equiv 0$ for all time. The equal time and the two time correlation functions are then defined by

$$D_{ij}(t)=\sum_{[\sigma]}\sigma_i\sigma_jP([\sigma],t), \tag{7}$$

$$C_{ij}(t,t')=\sum_{[\sigma][\sigma']}\sigma_i\sigma_j'P([\sigma'],t')P([\sigma'],t'|[\sigma],t) \tag{8}$$

with $t\geqslant t'$ and $C_{ij}(t,t)=D_{ij}(t)$. $P([\sigma'],t'|[\sigma],t)$ is the conditional probability to find the system in the configuration $[\sigma]$ at the time t, given that it was in the configuration $[\sigma']$ at the earlier time t'. We assume that space translation invariance holds at all times.

From Eqs. (5) and (7) one can show that the equal time correlation function satisfies the equation of motion

$$\frac{d}{dt}D_{ij}(t)=-2D_{ij}(t)+\frac{\gamma}{2}[D_{i,j-1}(t)+D_{i,j+1}(t)+D_{i-1,j}(t)+D_{i+1,j}(t)] \tag{9}$$

for $i\neq j$ and $(d/dt)D_{ii}(t)=0$, since $D_{ii}(t)=1$. Similarly, from Eq. (8) taking into account that also the conditional probability satisfies the master equation (5) one finds

$$\frac{\partial}{\partial t}C_{ij}(t)=-C_{ij}(t,t')+\frac{\gamma}{2}[C_{i-1,j}(t,t')+C_{i+1,j}(t,t')]. \tag{10}$$

The single spin conditional expectation can be computed explicitely obtaining [8]

$$\sum_{[\sigma]}P([\sigma'],t'|[\sigma],t)\sigma_i=\sum_l\sigma_l'F_{i-l}(t-t'), \tag{11}$$

where $F_{i-m}(t-t')=e^{-(t-t')}I_{i-m}[\gamma(t-t')]$ and $I_n(x)$ are the Bessel functions of imaginary argument. Then, using the definitions (7) and (8), the two times and the equal times correlation functions are related by

$$C_{ij}(t,t')=\sum_l D_{jl}(t')F_{i-l}(t-t') \tag{12}$$

or in Fourier space one finds

$$C_k(t,t')=D_k(t')e^{-\gamma_k(t-t')} \tag{13}$$

with $\gamma_k=1-\gamma\cos k$. Given the initial condition $D_{i,j}(0)$ Eq. (9) can be solved exactly [8]. After some microscopic time t_0, which we assume much smaller than $t_{\rm eq}$, memory of the initial condition is lost and for $k\ll 1$, $\xi_F\gg 1$ one has [9]

$$D_k(t')=2\left(\frac{t'}{\pi}\right)^{1/2}\frac{1}{k^2+\xi_F^{-2}}\int_0^1 dy\,y^{-1/2}\times[\xi_F^{-2}e^{-\xi_F^{-2}t'y}+k^2e^{(-[k^2+\xi_F^{-2}]t'+k^2t'y)}], \tag{14}$$

where we have expanded γ_k to lowest order in k and ξ_F^{-1} using $\gamma=1/\cosh(\xi_F^{-1})$. The form (14) for the equal time structure factor obeys the scaling relation $D_k(t')=(t')^{1/2}g(k^2t',t'/t_{\rm eq})$ with the limits

$$D_k(t')\sim\begin{cases}(t_{\rm eq})^{1/2}g_{\rm eq}(k^2t_{\rm eq}) & \text{for } t'/t_{\rm eq}\gg 1,\\ (t')^{1/2}g_{sc}(k^2t') & \text{for } t'/t_{\rm eq}\ll 1.\end{cases} \tag{15}$$

Inserting Eq. (14) into Eq. (13) and inverting the Fourier transform, the corresponding scaling form for the two time correlation function is obtained $C_{i,j}(t,t')=f(|i-j|/(t')^{1/2},t/t',t'/t_{\rm eq})$ with the limiting behaviors

$$C_{i,j}(t,t')\sim\begin{cases}f_{\rm eq}\left(\dfrac{|i-j|}{(t_{\rm eq})^{1/2}},\dfrac{t-t'}{t_{\rm eq}}\right) & \text{for } t'/t_{\rm eq}\gg 1,\\[2ex] f_{sc}\left(\dfrac{|i-j|}{(t')^{1/2}},\dfrac{t}{t'}\right) & \text{for } t'/t_{\rm eq}\ll 1.\end{cases} \tag{16}$$

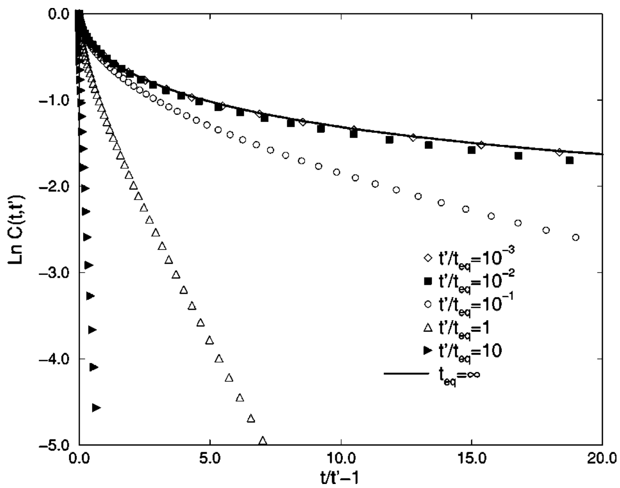

FIG. 1. Plot of the autocorrelation function for different values of $\tau = t'/t_{\rm eq}$ and $t_{\rm eq} = 10^3$. The continuous line is the plot of the anlytical solution (17) coresponding to $T_F = 0$.

The use of the small k approximation (14) is justified since for small enough T_F the structure factor builds up a large and narrow peak about $k=0$ which gives the main contribution to the integral over k. In the following it will be sufficient to consider the autocorrelation function ($i=j$). In the case of the zero temperature quench the scaling behavior at the bottom of Eq. (16) is obeyed for all times since $t_{\rm eq} = \infty$ and the explicit form of the scaling function is given by [9,10]

$$C_{i,i}(t,t') = f_{sc}(t/t') = \frac{2}{\pi} \arcsin \sqrt{\frac{2}{1+\frac{t}{t'}}}. \tag{17}$$

With $T_F > 0$ and $t_{\rm eq} < \infty$, $C_{i,i}(t,t')$ can be computed by a combination of analytical and numerical integration (see the Appendix). In Fig. 1 we have plotted $\log C_{i,i}(t,t')$ against $x = (t/t'-1)$ for different values of $\tau = t'/t_{\rm eq}$ illustrating the crossover from the scaling form (17) to the exponential decay corresponding to the top of Eq. (16) as τ grows from small to large values.

III. RESPONSE FUNCTION

As stated in the Introduction, let us now assume that after the quench to T_F, at some time $t_w > 0$, a site and time dependent external field $h_i(t)$ is switched on. We are interested in the response in the magnetization to the action of this field. More specifically, we wish to investigate the relation between the magnetization response and the correlation function in absence of the field.

For sufficiently small external field, the response in the magnetization at site i and $t > t_w$ is given by linear response theory

$$\begin{aligned}\Delta\langle \sigma_i(t)\rangle &= \langle \sigma_i(t)\rangle_h - \langle \sigma_i(t)\rangle_{h=0} \\ &= \sum_j \int_{t_w}^{t} dt' R_{i,j}(t,t') h_j(t'),\end{aligned} \tag{18}$$

where

$$R_{i,j}(t,t') = \left(\frac{\delta \langle \sigma_i(t)\rangle_h}{\delta h_j(t')} \right)_{h=0} \tag{19}$$

is the causal response function. The difference between the two expectation values in Eq. (18) is given by $\Delta\langle \sigma_i(t)\rangle = \Sigma_{[\sigma]} \sigma_i \Delta P([\sigma],t)$ where $\Delta P([\sigma],t) = P_h([\sigma],t) - P([\sigma],t)$ is the difference between the probabilities with and without the field. The time evolution of $P_h([\sigma],t)$ is given by the master equaltion (5) with the transition rate (6) replaced by $w_h(\sigma_i) = w(\sigma_i) + \Delta w(\sigma_i)$ where $\Delta w(\sigma_i) = -\tanh(h_i/T_F)\sigma_i w(\sigma_i)$. Taking h_i/T_F sufficiently small $\tanh(h_i/T_F) \simeq h_i/T_F$ and following [8] up to first order we have

$$\begin{aligned}\Delta P([\sigma],t) = \frac{1}{T_F} \sum_{[\sigma']} \sum_i \sigma_i' \int_{t_w}^{t} dt' h_i(t') [w(\sigma_i') P([\sigma'],t') \\ + w(-\sigma_i') P([R_i \sigma'],t')] P([\sigma'],t' | [\sigma],t)\end{aligned} \tag{20}$$

yielding

$$\begin{aligned}R_{i,j}(t,t') = \frac{1}{T_F} \sum_{[\sigma][\sigma']} \sigma_j' [w(\sigma_j') P([\sigma'],t') \\ + w(-\sigma_j') P([R_j \sigma'],t')] P([\sigma'],t' | [\sigma],t) \sigma_i .\end{aligned} \tag{21}$$

Performing the sum over $[\sigma]$ first and using Eq. (11) we find

$$\begin{aligned}R_{i,j}(t,t') = \frac{1}{T_F} \sum_{[\sigma']} \sum_l \sigma_j' \sigma_l' \{ w(\sigma_j') P([\sigma'],t') \\ + w(-\sigma_j') P([R_j \sigma'],t') \} F_{i-l}(t-t')\end{aligned} \tag{22}$$

and since only the term with $l=j$ survives in the summation we have

$$R_{i,j}(t,t') = \frac{1}{T_F} \left[1 - \frac{\gamma}{2} [D_{j,j-1}(t') + D_{j,j+1}(t')] \right] F_{i-j}(t-t'). \tag{23}$$

In order to recast this result in terms of $C_{i,j}(t,t')$ let us differentiate Eq. (12) with respect to the time arguments

$$\begin{aligned}\frac{\partial}{\partial t'} C_{i,j}(t,t') = \sum_l \frac{dD_{j,l}(t')}{dt'} F_{i-l}(t-t') \\ + \sum_l D_{j,l}(t') \frac{d}{dt'} F_{i-l}(t-t')\end{aligned} \tag{24}$$

$$\frac{\partial}{\partial t} C_{i,j}(t,t') = -\sum_l D_{j,l}(t') \frac{d}{dt'} F_{i-l}(t-t'). \tag{25}$$

Adding Eqs. (24) and (25) the summation in Eq. (25) cancels the second one in Eq. (24) yielding

$$\frac{\partial}{\partial t'} C_{i,j}(t,t') + \frac{\partial}{\partial t} C_{i,j}(t,t') = \sum_l \frac{dD_{j,l}(t')}{dt'} F_{i-l}(t-t'). \tag{26}$$

Next, using Eq. (9) for the time derivative when $l \neq j$, adding and subtracting a similar contribution with $l=j$ and using translational invariance we can rewrite

$$\begin{aligned} &\frac{\partial}{\partial t'} C_{i,j}(t,t') + \frac{\partial}{\partial t} C_{i,j}(t,t') \\ &= -2\sum_l \left\{ D_{j,l}(t') - \frac{\gamma}{2}[D_{j,l+1}(t') + D_{j,l-1}(t')] \right\} F_{i-l} \\ &\quad \times (t-t') + 2\left\{ D_{j,j}(t') - \frac{\gamma}{2}[D_{j,j+1}(t') + D_{j,j-1}(t')] \right\} \\ &\quad \times F_{i-j}(t-t'). \end{aligned} \tag{27}$$

Using Eqs. (12) and (10) the first sum in the right hand side is given by $2(\partial/\partial t) C_{i,j}(t,t')$, while the second term coincides, up to a constant factor, with the right hand side of Eq. (23), since $D_{j,j}(t')=1$. Therefore, we finally get the following expression for the response function:

$$R_{i,j}(t,t') = \frac{1}{2T_F}\left[\frac{\partial}{\partial t'} C_{i,j}(t,t') - \frac{\partial}{\partial t} C_{i,j}(t,t') \right] \tag{28}$$

which can be rewritten in the form (1)

$$R_{i,j}(t,t') = \frac{X_{i,j}(t,t')}{T_F} \frac{\partial}{\partial t'} C_{i,j}(t,t') \tag{29}$$

with

$$X_{i,j}(t,t') = \frac{1}{2}\left[1 - \frac{(\partial/\partial t) C_{i,j}(t,t')}{(\partial/\partial t') C_{i,j}(t,t')} \right]. \tag{30}$$

Alternatively, we can expose the deviation from FDT through an additive term

$$R_{i,j}(t,t') = \frac{1}{T_F} \frac{\partial}{\partial t'} C_{i,j}(t,t') - \frac{1}{2T_F} B_{i,j}(t,t') \tag{31}$$

with

$$B_{i,j}(t,t') = \frac{\partial}{\partial t'} C_{i,j}(t,t') + \frac{\partial}{\partial t} C_{i,j}(t,t'). \tag{32}$$

When time translation invariance holds we have either $X_{i,j}(t,t')=1$ or $B_{i,j}(t,t')=0$ and the usual FDT is recovered.

IV. RANDOM EXTERNAL FIELD

In the simulations [4] the external field is taken random with site independent bimodal distribution

$$P[h] = \Pi_i \left[\frac{1}{2}\delta(h_i - h) + \frac{1}{2}\delta(h_i + h) \right]. \tag{33}$$

The reason for this choice is not to bias the evolution toward the formation of predominantly positive or negative domains through the introduction of the external perturbation. The quantity of interest then is the staggered magnetization

$$M(t,t_w) = \frac{1}{N}\sum_i \overline{\Delta\langle \sigma_i(t)\rangle \frac{h_i}{h}}, \tag{34}$$

where the overbar represents the average over the field configurations. From Eqs. (18) and (33) the integrated response is given by

$$\chi_{ii}(t,t_w) = \frac{1}{h} M(t,t_w) = \int_{t_w}^{t} dt' R_{ii}(t,t'). \tag{35}$$

Dropping the double index and inserting the form (29) of the response function, if the FDR depends on time only through $C(t,t')$ we have

$$T_F \chi[C(t,t_w)] = \int_{C(t,t_w)}^{1} dC X(C), \tag{36}$$

namely, also the integrated response depends on time only through the autocorrelation function. This occurs when FDT holds with $X(C)=1$ yielding

$$T_F \chi[C(t,t_w)] = [1 - C(t,t_w)] \tag{37}$$

and when the scaling form (17) holds. In that case from Eq. (30) follows

$$X(t,t') = \frac{1}{2}\left[1 + \frac{t'}{t} \right] \tag{38}$$

and inverting Eq. (17)

$$\frac{t'}{t} = \frac{\sin^2\left[\frac{\pi}{2} C(t,t')\right]}{2 - \sin^2\left[\frac{\pi}{2} C(t,t')\right]} \tag{39}$$

we find

$$X(C) = \frac{1}{2 - \sin^2\left(\frac{\pi}{2} C\right)}. \tag{40}$$

Inserting this into Eq. (36) we obtain

$$T_F \chi[C(t,t_w)] = \sqrt{\frac{2}{\pi}} \arctan\left[\sqrt{2}\cot\left(\frac{\pi}{2} C(t,t_w) \right) \right]. \tag{41}$$

We have then proceeded to compute (see the Appendix) $T_F \chi(t,t_w)$ with the values of the parameters ($t_{\mathrm{eq}} = 10^3$) corresponding to the behavior of the autocorrelation function displayed in Fig. 1 and we have plotted $T_F \chi(t,t_w)$ against $C(t,t_w)$ in Fig. 2 for different values of t_w/t_{eq}. In order to understand the plot notice that if t_{eq} is finite, from Eq. (35) follows $\lim_{t\to\infty} T_F \chi(t,t_w) = (T_F/h) M_{\mathrm{eq}} = 1$ where M_{eq} is the equilibrium value of the magnetization. On the other hand, with a finite t_{eq} one has also $\lim_{t\to\infty} C(t,t_w) = 0$. Therefore,

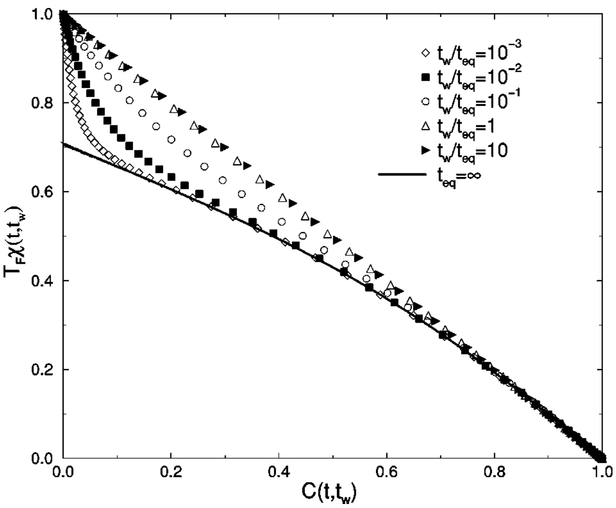

FIG. 2. Plot of the integrated response for different values of t_w/t_{eq} and $t_{\mathrm{eq}}=10^3$. The continuous line is the plot of the anlytical solution (41) coresponding to $T_F=0$.

when plotting $T_F\chi$ vs C all the curves starting out at ($C=1$, $T_F\chi=0$) must end up in the same point ($C=0$, $T_F\chi=1$). The dependence on t_w/t_{eq} enters on how the initial and the final point are joined. Thus, if $t_w/t_{\mathrm{eq}}>1$, FDT holds over the entire time interval (t_w,t) and the plot is linear according to Eq. (37). However, if $t_w/t_{\mathrm{eq}}<1$ then it is possible to have also $t/t_{\mathrm{eq}}<1$. In that case $C(t,t')$ obeys the scaling form (17) and in the range of values of C where this holds, $T_F\chi$ follows the shape (41). This forces the plot to fall below the straight line of the FDT, but eventually as C decreases the plot must raise again in order to reach the value $T_F\chi=1$ at $C=0$. Therefore, the peculiar shape of the curves displaying a change in concavity is a consequence of a finite equilibration time. The final upword bending of the curves corresponds to interrupted aging and that is where the curves do depend on t_w. Furthermore, the range of values where the plot follows the shape (41) is larger the smaller the value is of t_w/t_{eq}. In the limiting case $t_{\mathrm{eq}}=\infty$ aging holds for all time and the plot obeys Eq. (41) over the entire range of C values.

V. CONCLUSIONS

The relaxation dynamics of the one-dimensional Ising model allows us to analyze in detail the transition from the off equilibrium to the equilibrium regime. In particular, we have obtained the crossover in the FDR from the nontrivial form $X(C)$ given by Eq. (40) to $X(t,t')\equiv 1$ as a manifestation at the level of the response function of the crossover in the underlying correlation function from aging to time translation invariance.

A comment should be made about the shape of $X(C)$. In the case of the zero temperature quench (40) holds for all time. On the other hand, the zero temperature quench is a phase ordering process eventually leading to the coexistence of ordered phases as in the quench below the critical point of a system with a finite critical temperature. In the latter case $X(C)$ displays [4,5] a qualitatively different behavior decreasing from 1 and flattening to zero, while in our case $X(C)$ decreases from 1 toward 1/2 as C goes to zero. Although we do not have a complete understanding of the origin of this discrepancy, we believe this to be related to the absence in the one-dimensional case of the asymmetry term in the relation between $R(t,t')$ and $C(t,t')$. In the context of Langevin dynamics one can derive [11] in full generality

$$R(t,t')=\frac{1}{2T_F}\left(\frac{\partial}{\partial t'}-\frac{\partial}{\partial t}\right)C(t,t')-\frac{1}{2T_F}A(t,t'), \quad (42)$$

where $A(t,t')$ is the asymmetry term which vanishes for linear dynamics. From Eq. (42) the FDR takes the following general form:

$$X(t,t')=\frac{1}{2}\left[1-\frac{\frac{\partial}{\partial t}C(t,t')}{\frac{\partial}{\partial t'}C(t,t')}\right]-\frac{1}{2}\,\frac{A(t,t')}{\frac{\partial}{\partial t'}C(t,t')} \quad (43)$$

and if we assume scaling $C(t,t')=f(t/t')$ the square brackets contribution is given by Eq. (38) independently from the form of $f(x)$. Therefore, if we accept that X is a function of C when scaling holds, in order to have $\lim_{C\to 0}X(C)<1/2$ the asymmetry term must necessarily be nonzero. Now, from Eq. (28) follows that in the one-dimensional Ising model with Glauber dynamics the asymmetry is absent. Indeed, in this case as Eqs. (9) and (10) show, dynamics is linear. Another example of linear dynamics leading to $\lim_{C\to 0}X(C)=1/2$ is the massless Gaussian model [11]. Conversely, if one considers the Ising model with higher dimensionality and a finite critical temperature, the equations of motion for the two point correlation functions are coupled to higher correlation functions producing a nonlinearity which in turn is expected to produce a nonvanishing asymmetry in the off equilibrium regime.

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APPENDIX

In order to carry out the computation of $T_F\chi(t,t_w)$ we start from the sum of the time derivatives of $C_k(t,t')$ obtained from Eq. (13)

$$\left(\frac{\partial}{\partial t'}+\frac{\partial}{\partial t}\right)C_k(t,t')=\frac{dD_k(t')}{dt'}e^{-\gamma_k(t-t')}. \quad (A1)$$

Using Eq. (14) for $D_k(t')$ and carrying out integrations by parts we find

$$\frac{dD_k(t')}{dt'}=\frac{2}{\sqrt{\pi t'}}e^{-\xi_F^{-2}t'}-2\sqrt{\frac{t'}{\pi}}e^{-(k^2+\xi_F^{-2})t'}k^2 \times\int_0^1\frac{dy}{\sqrt{y}}e^{k^2t'y}. \quad (A2)$$

Inserting this result into Eq. (A1) and integrating over k we have

$$B(t,t')=\left(\frac{\partial}{\partial t'}+\frac{\partial}{\partial t}\right)C(t,t')$$

$$=\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{dD_k(t')}{dt'}e^{-(1/2)(k^2+\xi_F^{-2})(t-t')}$$

$$=\frac{1}{\pi}\sqrt{\frac{2}{t'(t-t')}}e^{-(1/2\xi_F^{-2}(t+t')}$$

$$-\frac{1}{\pi}e^{-(1/2)\xi_F^{-2}(t+t')}\int_0^1 dy\sqrt{\frac{2t'}{y(t+t'-2yt')^3}}$$

$$=\frac{1}{\pi(t+t')}\sqrt{\frac{2(t-t')}{t'}}e^{-(1/2)\xi_F^{-2}(t+t')}. \tag{A3}$$

Next, Fourier transforming the equation of motion (9) one obtains

$$\frac{d}{dt}D_k(t)=-2\gamma_k D_k(t)+r(t) \tag{A4}$$

with

$$r(t)=\frac{e^{-(1/2)\xi_F^{-2}t}}{\sqrt{\pi t}}+\xi_F^{-1}Erf(\sqrt{\xi_F^{-2}t}), \tag{A5}$$

where *Erf* is the error function. Inserting Eq. (A4) in the right hand side of Eq. (A1) and using $(\partial/\partial t)C_k(t,t')=-\gamma_k C_k(t,t')$ one finds

$$\left(\frac{\partial}{\partial t'}-\frac{\partial}{\partial t}\right)C_k(t,t')=r(t')e^{-(1/2)(k^2+\xi_F^{-2})(t-t')} \tag{A6}$$

which after integration over k gives

$$\left(\frac{\partial}{\partial t'}-\frac{\partial}{\partial t}\right)C(t,t')=\frac{2}{\pi}\frac{e^{-(1/2)\xi_F^{-2}(t+t')}}{\sqrt{2t'(t-t')}}+\sqrt{\frac{2}{\pi}}\xi_F^{-1}\frac{\mathrm{Erf}(\sqrt{\xi_F^{-2}t'})}{\sqrt{t-t'}}\times e^{-(1/2)\xi_F^{-2}(t-t')}. \tag{A7}$$

Inserting this result in Eq. (28) with $i=j$ the integrated response $T_F\chi(t,t_w)$ is obtained carrying out numerically the integration in Eq. (35). Similarly, the autocorrelation function is obtained by taking the difference of Eqs. (A3) and (A7) and carrying out numerically the time integration in

$$C(t,t_w)=1+\int_{t_w}^{t}ds\frac{\partial}{\partial s}C(s,t_w). \tag{A8}$$

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